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Multigrid Methods for Combined Finite Difference and Fourier Problems

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Multigrid methods for combined finite difference and Fourier pseudospectral problems are considered. Applications to two- and three-dimensional elliptic systems are given. Suitable relaxation schemes for isotropic and anisotropic problems are presented. Numerical results demonstrate the efficiency of these methods. © 1988 Academic Press, Inc.

1. INTRODUCTION

We study elliptic problems with combined Dirichlet (or Neumann) and periodic boundary conditions. The Dirichlet direction is discretized by means of finite differences and periodicity is enforced by a Fourier pseudospectral method. A convergence analysis of these method was already given by Canuto, Maday, and Quarteroni [5].

Spectral methods [13, 15, 16] usually employ Chebyshev polynomials in the nonperiodic direction. This leads to highly accurate approximations with relatively few degrees of freedom. A disadvantage of this approach is the high condition number which increases as N^4 (N: number of grid points) (see Zang, Wong, and Hussaini [18]). Furthermore, the use of Gauss (-Lobatto) nodes is necessary in order to avoid a significant deterioration of the discretization error (see [4, 6, 7]). Equidistant nodes, for instance, would lead to an exponential perturbation of convergence (see, e.g., De Boor [9]). For employing fast Fourier transforms (FFTs) the Chebyshev nodes are commonly used.

For the finite difference discretization we attain a condition number of N^2 and, in addition, equidistant nodes are taken which give uniformly distributed approximation values over the whole region.

Multigrid methods for the solution of spectral systems were introduced by Zang *et al.* [18, 19]. Brandt, Fulton, and Taylor [3] have improved them for periodic problems. Erlebacher, Zang, and Hussaini [10] investigated the specific problems arising from three-dimensional (Fourier-) Helmholtz equations. We achieved some improvements for Dirichlet problems [13]. The experiences concerning suitable relaxation techniques are now adopted to combined finite difference and Fourier spectral problems.

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In Section 3 we first consider isotropic problems where weighted residual relaxation schemes [2] yield greatly improved smoothing rates. Here the optimal weights are calculated for two- and three-dimensional elliptic model problems. Although the smoothing analysis is rigorous only for constant coefficients the rates obtained even hold for variable coefficients.

For anisotropic problems or when different step sizes (for each direction) are used, pointwise relaxation is inefficient and linewise relaxation techniques are necessary. Here we employ a zebra line relaxation (see [3, 13, 17]) for defect correction. In Section 4 we describe the efficient implementation of these methods. Furthermore different types of Richardson relaxation are compared. Numerical results for some examples introduced by Street, Zang, and Hussaini [16] show the efficiency of multigrid methods for combined finite difference and Fourier problems.

2. Finite Difference and Fourier Discretization

We consider two- and three-dimensional elliptic problems, given as

$$(au_x)_x + (bu_y)_y = f \qquad \text{on} \quad \Omega = (0, 2\pi)^2 \tag{2.1}$$

and

$$(au_x)_x + (bu_y)_y + (cu_z)_z = f \qquad \text{on} \quad \Omega = (0, 2\pi)^3 \tag{2.2}$$

with periodic or Dirichlet (or Neumann) boundary conditions in x, y, z. Here a, b, c, f denote given functions, defined on Ω. For anisotropic problems we use the interval $(-1, 1)$ in the direction of Dirichlet boundary conditions instead of $(0, 2\pi)$.

The Dirichlet directions are discretized by usual second (or fourth) order finite differences (FD) and in the direction of periodicity we use a pseudospectral Fourier (PSF) method. The essence of the PSF method is to approximate the solution u by a truncated Fourier series which satisfies (2.1) or (2.2) at a finite set of discrete points known as collocation points. The collocation points for the PSF method are

$$y_j = \frac{j2\pi}{N} \qquad (j = 0, ..., N-1)$$

and the points for the FD discretization are

$$x_i = \frac{i2\pi}{M} \qquad \text{on } (0, 2\pi) \qquad \left(\text{resp. } x_i = \frac{2i}{M} - 1 \text{ on } (-1, 1)\ (i = 1, ..., M-1)\right).$$

Here M and N are chosen as certain powers of 2. The five-point second-order FD operator applied to the one-dimensional operator $(au')'$ at $x = x_i$ is given as

$$\frac{1}{h_M^2}\left(a(x_{i-1/2})\, u^h(x_{i-1}) - (a(x_{i-1/2}) + a(x_{i+1/2}))\, u^h(x_i) + a(x_{i+1/2})\, u^h(x_{i+1})\right), \tag{2.3}$$

where $h_M = 2\pi/M$ on $(0, 2\pi)$ (resp. $h_M = 2/M$ on $(-1, 1)$). For fourth-order FD applied to u'' we use

$$\frac{1}{12h_M^2}(u^h(x_{i-2}) + u^h(x_{i+2}) - 16(u^h(x_{i-1}) + u^h(x_{i+1})) + 30u^h(x_i)). \tag{2.4}$$

At the points next to the boundary we employ the second-order FD (2.3) (see Hackbusch [12]). By monotonicity arguments it can be shown that this combination of (2.3), (2.4) yields fourth-order convergence. Other higher order FD operators as, e.g., the "Mehrstellen"-operator (see, e.g., Collatz [8]) are not applicable since not all directions are discretized by FD methods.

For Neumann boundary conditions at $x = x_0$ we use first- and second-order FDs, i.e., $u'(x_0)$ is discretized by

$$\frac{1}{h_M}(u^h(x_1) - u^h(x_0)) \tag{2.5}$$

and

$$\frac{1}{h_M}(-1.5u^h(x_0) + 2u^h(x_1) - 0.5u^h(x_2)). \tag{2.6}$$

Here (2.5) is only used for preconditioning and (2.6) for evaluating the residual.

In the direction of periodicity we use the "midpoint discretization," recommended by Brandt *et al.* [3]. By this approach the information in the highest mode is retained and an improved accuracy can be achieved.

3. Isotropic Problems

The problem (2.1) (resp. (2.2)) is called isotropic if $a = b$ (resp. $a = b = c$) on Ω. The corresponding discrete problems are called isotropic if, additionally, $M = N$, which will be assumed throughout this section. For such problems the effectiveness of the residual can be increased by updating the approximation u^h by a suitable weighted sum of values of the residual r^h at the neighbouring points (see Brandt *et al.* [1, 3]). This technique is usually referred to as the weighted residual relaxation scheme.

For problem (2.1) and a fixed point (x_j, y_k) the scheme for calculating the new approximation $\bar{u}^h$ is given by

$$\begin{aligned}\bar{u}^h_{j,k} = u^h_{j,k} - \omega[&\alpha r^h_{j,k} + \beta_x(r^h_{j-1,k} + r^h_{j+1,k}) + \beta_y(r^h_{j,k-1} + r^h_{j,k+1}) \\ &+ \gamma(r^h_{j-1,k-1} + r^h_{j-1,k+1} + r^h_{j+1,k-1} + r^h_{j+1,k+1})]\end{aligned}$$

with the relaxation parameter ω and weights α, β_x, β_y, γ.

For problem (2.2) and a fixed point (x_j, y_k, z_l) it is given by

$$\begin{aligned}\bar{u}^h_{j,k,l} = u^h_{j,k,l} - \omega[&\alpha r^h_{j,k,l} + \beta_x(r^h_{j-1,k,l} + r^h_{j+1,k,l}) \\ &+ \beta_y(r^h_{j,k-1,l} + r^h_{j,k+1,l}) + \beta_z(r^h_{j,k,l-1} + r^h_{j,k,l+1}) \\ &+ \gamma_{xy}(r^h_{j-1,k-1,l} + r^h_{j-1,k+1,l} + r^h_{j+1,k-1,l} + r^h_{j+1,k+1,l}) \\ &+ \gamma_{xz}(r^h_{j-1,k,l-1} + r^h_{j-1,k,l+1} + r^h_{j+1,k,l-1} + r^h_{j+1,k,l+1}) \\ &+ \gamma_{yz}(r^h_{j,k-1,l-1} + r^h_{j,k-1,l+1} + r^h_{j,k+1,l-1} + r^h_{j,k+1,l+1}) \\ &+ \delta(r^h_{j-1,k-1,l-1} + r^h_{j-1,k+1,l-1} + r^h_{j+1,k-1,l-1} + r^h_{j+1,k+1,l-1} \\ &+ r^h_{j-1,k-1,l+1} + r^h_{j-1,k+1,l+1} + r^h_{j+1,k-1,l+1} + r^h_{j+1,k+1,l+1})].\end{aligned}$$

3.1. *Smoothing Analysis*

For calculating the optimal weights we use local mode analysis [2]. The coefficients a, b, c are considered to be constant. The discrete eigenfunctions for the FD (resp. PSF) method are now given by

$$\varphi^{h,n}_j = \sin\left(\frac{n}{2}x_j\right) \qquad (x_j = jh_N; j = 1, ..., N-1)$$

for $n = 1, ..., N-1$ (resp.

$$\psi^{h,m}_k = \exp(imy_k) \qquad (y_k = k2\pi/N; k = 0, ..., N-1)$$

for $m = -N/2, ..., N/2$).

With $\vartheta_x = n_x\pi/N$ $(n_x = 0, ..., N)$ (respp. $\vartheta_y = n_y 2\pi/N$ $(n_y = -N/2, ..., N/2)$) we introduce the corresponding discrete wavenumbers. We choose the relaxation parameter ω as

$$\omega = \frac{4\pi^2}{aN^2} = \frac{4\pi^2}{bN^2} = \frac{4\pi^2}{cN^2}. \tag{3.1}$$

Depending on the combination of second-order FD and PSF, different convergence factors are available. (If the error before relaxation has a component $A_\vartheta E_\vartheta$, the error after relaxation will have the component $\bar{A}_\vartheta E_\vartheta$. The quotient $\bar{A}_\vartheta/A_\vartheta$ is called the convergence factor $\mu(\vartheta)$.) We get for problem (2.1) and the combination

$$\begin{aligned}\text{PSF/PSF: } &\mu(\vartheta) = 1 - (\vartheta_x^2 + \vartheta_y^2)\,\chi_{xy} \\ \text{FD/PSF: } &\mu(\vartheta) = 1 - (2(1 - \cos(\vartheta_x)) + \vartheta_y^2)\,X_{xy} \\ \text{FD/FD: } &\mu(\vartheta) = 1 - 2(2 - \cos(\vartheta_x) - \cos(\vartheta_y))\,X_{xy},\end{aligned}$$

where $X_{xy} = \alpha + 2\beta_x\cos(\vartheta_x) + 2\beta_y\cos(\vartheta_y) + 4\gamma\cos(\vartheta_x)\cos(\vartheta_y)$. We further get for problem (2.2) and the combination

$$\text{PSF/PSF/PSF: } \mu(\vartheta) = 1 - (\vartheta_x^2 + \vartheta_y^2 + \vartheta_z^2)\, X_{xyz}$$
$$\text{FD/PSF/PSF: } \mu(\vartheta) = 1 - (2(1 - \cos(\vartheta_x)) + \vartheta_y^2 + \vartheta_z^2)\, X_{xyz}$$
$$\text{FD/FD/PSF: } \mu(\vartheta) = 1 - (2(2 - \cos(\vartheta_x) - \cos(\vartheta_y)) + \vartheta_z^2)\, X_{xyz}$$
$$\text{FD/FD/FD: } \mu(\vartheta) = 1 - 2(3 - \cos(\vartheta_x) - \cos(\vartheta_y) - \cos(\vartheta_z))\, X_{xyz},$$

where

$$\begin{aligned} X_{xyz} = {} & \alpha + 2\beta_x \cos(\vartheta_x) + 2\beta_y \cos(\vartheta_y) + 2\beta_z \cos(\vartheta_z) \\ & + 4\gamma_{xy} \cos(\vartheta_x)\cos(\vartheta_y) + 4\gamma_{xz} \cos(\vartheta_x)\cos(\vartheta_z) \\ & + 4\gamma_{yz} \cos(\vartheta_y)\cos(\vartheta_z) + 8\delta \cos(\vartheta_x)\cos(\vartheta_y)\cos(\vartheta_z). \end{aligned}$$

The wavenumbers $\vartheta_x, \vartheta_y, \vartheta_z$ are discrete values but for a smoothing analysis we consider them continuously distributed; i.e., the modes representable on the fine grid have $|\vartheta| = \max\{|\vartheta_x|, |\vartheta_y|, |\vartheta_z|\} \leqslant \pi$. The modes not representable on the coarse grids (with step size $2\pi/\rho N$ ($\rho < 1$)) are the high wavenumbers with $|\vartheta| > \rho\pi$. Including $|\vartheta| = \rho\pi$ as the high wavenumber, the smoothing factor is defined as

$$\bar{\mu} = \max\{|\mu(\vartheta)| : \rho\pi \leqslant |\vartheta| \leqslant \pi\}.$$

It is obvious that $\bar{\mu}$ and the optimal parameters do not depend on N. In the case of the single parameter α (i.e., the other parameters are equal to zero) the weighted residual scheme reduces to the usual Richardson relaxation scheme [18]. Only in this case can the optimal α and corresponding $\bar{\mu}$ be calculated analytically and we get for problem (2.1):

$$\text{PSF/PSF (see [3]): } \alpha = \frac{2}{(2+\rho^2)\,\pi^2}, \qquad \bar{\mu} = \frac{2-\rho^2}{2+\rho^2}$$

$$\text{FD/PSF: } \alpha = \frac{2}{6+\pi^2-2\cos\rho\pi}, \qquad \bar{\mu} = \frac{2+\pi^2+2\cos\rho\pi}{6+\pi^2-2\cos\rho\pi}$$

$$\text{FD/FD (see [1]): } \alpha = \frac{1}{5-\cos\rho\pi}, \qquad \bar{\mu} = \frac{3+\cos\rho\pi}{5-\cos\rho\pi}.$$

For problem (2.2) we get

$$\text{PSF/PSF/PSF (see [10]): } \alpha = \frac{2}{(3+\rho^2)\,\pi^2}, \qquad \bar{\mu} = \frac{3-\rho^2}{3+\rho^2}$$

$$\text{FD/PSF/PSF: } \alpha = \frac{1}{3+\pi^2-\cos\rho\pi}, \qquad \bar{\mu} = \frac{1+\pi^2+\cos\rho\pi}{3+\pi^2-\cos\rho\pi}$$

$$\text{FD/FD/PSF: } \alpha = \frac{2}{10+\pi^2-2\cos\rho\pi}, \qquad \bar{\mu} = \frac{6+\pi^2+2\cos\rho\pi}{10+\pi^2-2\cos\rho\pi}$$

$$\text{FD/FD/FD (see [1]): } \alpha = \frac{1}{7-\cos\rho\pi}, \qquad \bar{\mu} = \frac{5+\cos\rho\pi}{7-\cos\rho\pi}.$$

The standard mesh ratio is $\rho = \frac{1}{2}$, where we obtain

$$\text{PSF/PSF:} \quad \alpha = \frac{8}{9\pi^2}, \qquad \bar{\mu} = \frac{7}{9} \doteq 0.7778$$

$$\text{FD/PSF:} \quad \alpha = \frac{2}{6+\pi^2}, \qquad \bar{\mu} = \frac{2+\pi^2}{6+\pi^2} \doteq 0.7479$$

$$\text{FD/FD:} \quad \alpha = \frac{1}{5}, \qquad \bar{\mu} = \frac{3}{5} = 0.6$$

$$\text{PSF/PSF/PSF:} \quad \alpha = \frac{8}{13\pi^2}, \qquad \bar{\mu} = \frac{11}{13} \doteq 0.8462$$

$$\text{FD/PSF/PSF:} \quad \alpha = \frac{1}{3+\pi^2}, \qquad \bar{\mu} = \frac{1+\pi^2}{3+\pi^2} \doteq 0.8446$$

$$\text{FD/FD/PSF:} \quad \alpha = \frac{2}{10+\pi^2}, \qquad \bar{\mu} = \frac{6+\pi^2}{10+\pi^2} \doteq 0.7987$$

$$\text{FD/FD/FD:} \quad \alpha = \frac{1}{7}, \qquad \bar{\mu} = \frac{5}{7} \doteq 0.7143.$$

If more than one parameter is considered the optimal weights and $\bar{\mu}$ must be computed numerically. For this purpose we employ the differential-correction algorithm [14]. Tables I and II contain the results for problems (2.1) and (2.2). It becomes obvious that the weighted residual relaxation scheme improves the smoothing factor dramatically and the computational effort needed is relatively small. Further numerical tests show that the introduction of more surrounding points leads to only small reductions in the smoothing factor.

TABLE I

Optimal Parameters and Smoothing Factors for Problem (2.1)

Method	α	β_x	β_y	γ	$\bar{\mu}$
PSF/PSF	0.9006 (−1)	0	0	0	0.7778
	0.1491	0.3024 (−1)	0.3024 (−1)	0	0.4718
	0.2240	0.7000 (−1)	0.7000 (−1)	0.2800 (−1)	0.1058
FD/PSF	0.1260	0	0	0	0.7479
	0.2311	0.2097 (−1)	0.6935 (−1)	0	0.3262
	0.2554	0.5483 (−1)	0.8694 (−1)	0.2423 (−1)	0.1414
FD/FD	0.2	0	0	0	0.6
	0.2927	0.4878 (−1)	0.4878 (−1)	0	0.2195
	0.2889	0.6567 (−1)	0.6567 (−1)	0.1970 (−1)	0.1595

TABLE II

Optimal Parameters and Convergence Factors for Problem (2.2)

Method	Parameters	Optimal values	Smoothing rate $\bar{\mu}$
PSF/PSF/PSF	α	0.6235 (−1)	0.8462
	α	0.1187	
	$\beta_x, \beta_y, \beta_z$	0.1777 (−1)	0.6417
	α	0.1245	
	$\beta_x, \beta_y, \beta_z$	0.3034 (−1)	
	$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	0.8950 (−1)	0.4732
	α	0.1472	
	$\beta_x, \beta_y, \beta_z$	0.4438 (−1)	
	$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	0.1926 (−1)	
	δ	0.8998 (−2)	0.1898
FD/PSF/PSF	α	0.7770 (−1)	0.8446
	α	0.1349	
	β_x	0.1278 (−1)	
	β_y, β_z	0.2319 (−1)	0.6063
	α	0.1665	
	β_x	0.2220 (−1)	
	β_y, β_z	0.4851 (−1)	
	γ_{xy}, γ_{xz}	0.6910 (−2)	
	γ_{yz}	0.1843 (−1)	0.3534
	α	0.1623	
	β_x	0.3454 (−1)	
	β_y, β_z	0.5031 (−1)	
	γ_{xy}, γ_{xz}	0.1561 (−1)	
	γ_{yz}	0.2187 (−1)	
	δ	0.7325 (−2)	0.2046
FD/FD/PSF	α	0.1007	0.7937
	α	0.1768	
	β_x, β_y	0.1667 (−1)	
	β_z	0.4263 (−1)	0.5621
	α	0.1964	
	β_x, β_y	0.3537 (−1)	
	β_z	0.5944 (−1)	
	γ_{xy}	0.6258 (−2)	
	γ_{xz}, γ_{yz}	0.1399 (−1)	0.3037
	α	0.1795	
	β_x, β_y	0.3863 (−1)	
	β_z	0.5684 (−1)	
	γ_{xy}	0.1251 (−1)	
	γ_{xz}, γ_{yz}	0.1729 (−1)	
	δ	0.5951 (−2)	0.2157
FD/FD/FD	α	0.1429	0.7143
	α	0.2384	
	$\beta_x, \beta_y, \beta_z$	0.3206 (−1)	0.4472
	α	0.2154	
	$\beta_x, \beta_y, \beta_z$	0.4387 (−1)	
	$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	0.1035 (−1)	0.3099
	α	0.1965	
	$\beta_x, \beta_y, \beta_z$	0.3975 (−1)	
	$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	0.1157 (−1)	
	δ	0.3694 (−2)	0.1967

Finally, we remark that in the case of different step sizes ($M \gg N$) or different lengths of intervals (e.g., $(-1, 1) \times (0, 2\pi)$) the smoothing factors deteriorate significantly. If we consider $M = N$ but $(-1, 1) \times (0, 2\pi)$ for the FD/PSF) method we obtain only a rate $\bar{\mu} \doteq 0.6735$ (for four weights). Hence relaxation schemes used for anisotropic problems (see Sect. 4) are appropriate smoothers.

3.2. *Implementation and Numerical Results*

We use a V-cycle with four grids (4, 8, 16, and 32 points in each direction). Fixed numbers $N_d = 1$ and $N_u = 0$ of relaxation steps on each grid in the downward and upward branches are employed. For grid transfers in the Fourier direction we use the usual Fourier interpolations [18]. In the direction of FDs we use, for restriction, the full weighting operator $[\frac{1}{4}\,\frac{1}{2}\,\frac{1}{4}]_N^{N/2}$ and for interpolation the linear interpolation operator $]\frac{1}{2}\ 1\ \frac{1}{2}[_{N/2}^{N}$, where $]\ [_{N/2}^{N}$ means that only the (surrounding) points which belong to both the coarse and the fine grid are taken into account (see [17]). We use the weighted residual relaxation, where the relaxation parameter ω is evaluated pointwise as suggested by the smoothing analysis (see (3.1)). The convergence factor ρ of the whole multigrid procedure is calculated by means of the power method [11]. The convergence factor per work unit is defined as $\rho_w = \rho^{1/W}$, where W denotes the work for relaxation on the different grids. The standard work unit is the work involved in one relaxation sweep on the finest grid. Hence we get, in the above situation, $W = 1.328125\ (N_d + N_u) = 1.328125$.

We consider problem (2.1) with variable coefficients

$$a(x, y) = b(x, y) = 1 + \varepsilon \exp(\cos(\beta(x + y))),$$

$\beta = 10\varepsilon$, $\varepsilon \in \{0.\ , 0.1, 0.2\}$.

We examine the FD/PSF method with 1, 2, and 4 weights, where the optimal values are taken from Table I. Numerical results are presented in Table III. It can be seen that for $\varepsilon = 0.2$ the results degrade somewhat. This becomes still more significant for larger ε.

TABLE III

Convergence Factors ρ/ρ_W for the FD/PSF Method

Number of parameters	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$
1	0.7381/ 0.7956	0.7419/ 0.7988	0.7437/ 0.8001
3	0.3065/ 0.4105	0.3098/ 0.4138	0.4717/ 0.5679
4	0.1981/ 0.2955	0.2005/ 0.2983	0.3878/ 0.4901

But this effect could be expected since the local mode analysis is only rigorous in the case of constant coefficients. It becomes obvious that the convergence factor ρ (and not ρ_w as for spectral multigrid methods) is close to the smoothing factor $\bar{\mu}$. In the case of 4 weights it even turns out to be something worse than $\bar{\mu}$. This behaviour is typical for FD multigrid methods (see [17]) and is due to the fact that the smoothing effect is not fully exploited as the reduction of low error frequencies by one coarse grid correction step is not good enough. Furthermore, the smoothing effect is even partly destroyed by the coarse grid correction which introduces new high frequencies by itself. This is also the reason why further smoothing steps are not profitable when the smoothing rate is small.

4. Anisotropic Problems

We consider anisotropic problems given as

$$(au_x)_x + (bu_y)_y = f \qquad \text{on} \quad (-1, 1) \times (0, 2\pi). \tag{4.1}$$

The discretuzation is done by means of the FD/PSF method. The coefficients a and b are not necessarily equal and the step size in x is assumed to be much smaller than the step size in y, i.e., $M \gg N$.

For such problems we prefer a relaxation scheme based on defect correction with the five-point second-order operator

$$\begin{aligned}
&\frac{M^2}{4}\,(a(x_{j-1/2}, y_k)\, u^h_{j-1,k} - (a(x_{j-1/2}, y_k) + a(x_{j+1/2}, y_k))\, u^h_{j,k} \\
&\qquad + a(x_{j+1/2}, y_k)\, u^h_{j+1,k}) \\
&\qquad + \frac{N^2}{4\pi^2}\,(b(x_j, y_{k-1/2})\, u^h_{j,k-1} \\
&\qquad - (b(x_j, y_{k-1/2}) + b(x_j, y_{k+1/2}))\, u^h_{j,k} + b(x_j, y_{k+1/2})\, u^h_{j,k+1}).
\end{aligned} \tag{4.2}$$

4.1. *Smoothing Analysis and Implementation*

An appropriate smoother for anisotropic problems is the line relaxation (see [3, 13]). We employ line relaxation for defect correction and, by making one sweep, we obtain an approximation from an initial approximation of zero.

We relax along lines of constant y by solving

$$\begin{aligned}
&\frac{M^2}{4}\, a(x_{j-1/2}, y_k)\, \bar{v}^h_{j-1,k} - \left(\frac{M^2}{4}\,(a(x_{j-1/2}, y_k) + a(x_{j+1/2}, y_k))\right. \\
&\qquad \left. + \frac{N^2}{4\pi^2}\,(b(x_j, y_{k-1/2}) + b(x_j, y_{k+1/2}))\right) \bar{v}^h_{j,k} + \frac{M^2}{4}\, a(x_{j+1/2}, y_k)\, \bar{v}^h_{j+1,k} \\
&\quad = r^h_{j,k} - \frac{N^2}{4\pi^2}\,(b(x_j, y_{k-1/2})\, v^h_{j,k-1} + b(x_j, y_{k+1/2})\, v^h_{j,k+1})
\end{aligned}$$

for $\bar{v}^h$. Here r^h denotes the residual.

By solving first for the odd ("white") lines and then for the even ("black") lines we attain zebra line relaxation. If we relax along lines of constant y and then by doing an analogous sweep along lines of constant x we attain the alternating zebra line relaxation. The linear systems involved are symmetric and tridiagonal and were solved using a Cholesky decomposition. By computing the factorizations once and storing them, the computational costs per sweep could be cut in half.

We consider examples where $a \geqslant b$ on $(-1, 1) \times (0, 2\pi)$ and since, further, $M^2/4 \gg N^2/4\pi^2$, the x-direction is dominant and an exclusive smoothing in this direction is enough. A further step in the y-direction yields only a small improvement.

A smoothing analysis as done in [3, 13] shows that after the correction the frequencies with eigenvalues in $[1, \pi^2/4]$ have to be smoothed. The following choice of relaxation parameters is based on this prediction. Here the interval $[1, \pi^2/4]$ results from an asymptotic consideration, i.e., for large M, N. For small M, N the greatest eigenvalues are—dependent on M, N—somewhat smaller and somewhat better smoothing rates can be expected (see also Tables VII, VIII). Furthermore we compare three types of relaxation, stationary Richardson (SR), nonstationary Richardson (NSR) and minimal residual Richardson (MRR) relaxation. These types are already extensively investigated in the spectral literature (see also [13]). We use $N_d = 2$, $N_u = 0$ for SR, MRR and $N_d = 3$, $N_u = 0$ for NSR relaxation. The transfer operators are chosen as for isotropic problems. The multigrid structure consists of a V-cycle with four grids (16, 32, 64, 128 points in the x-direction and 4, 8, 16, 32 points in the y-direction).

4.2. *Numerical Results*

We tested the multigrid procedure for examples with coefficient functions

$$a(x, y) = b(x, y) = 1 + \varepsilon \exp(\cos(\beta(\pi x + y))) \tag{4.3}$$

and

$$a(x, y) = r(x)\,\alpha(x, y),\ b(x, y) = \frac{1}{r(x)}\,\alpha(x, y), \tag{4.4}$$

where $r(x) = 3 + 2x$ and

$$\alpha(x, y) = 1 + \varepsilon \exp(\cos(\beta(r(x) + y)))$$

for $\beta = 10\varepsilon$, $\varepsilon \in \{0.\ , 0.1, 0.2\}$.

These coefficients were introduced by Streett *et al.* [16] and were also examined in [13]. For (4.4) we consider Dirichlet and Neumann boundary conditions (at $x = -1$). The Neumann boundary conditions are treated explicitly and are discretized by the second-order FD (2.6). The corresponding preconditioning is done by first- or second-order FDs. In Tables IV–VI we present the convergence factors ρ_w for the above tests examples.

TABLE IV

Convergence Factors ρ_W for Coefficients (4.3)

Relaxation	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$
SR	0.5204	0.5245	0.5250
NSR	0.4131	0.4134	0.4139
MRR	0.4722	0.4724	0.4793

The numerical results show that the factors hardly deteriorate for increasing ε. This is due to the fact that the difference operator itself is preconditioned. As expected, the NSR and MRR relaxation yields better results than the SR relaxation. For the Dirichlet problems NSR is somewhat better than MRR. This confirms that the smoothing analysis has given a good prediction for the choice of relaxation parameters.

For the first-order preconditioning of Neumann boundary conditions the convergence factors turn out to be somewhat worse (see Table VI). On the other hand, for the second-order preconditioning the systems deviate (in one component) from tridiagonal systems. Nevertheless, we recommend this form since the corresponding elimination is not very costly. For the Chebyshev–Fourier discretization [13] of elliptic problems with Neumann boundary conditions the MRR relaxation was badly convergent with convergence factors equal to one. This problem does not arise for the FD/PSF method since the symmetric part of the discretization matrix is (negative) definite.

For some examples introduced by Streett *et al.* [16] we investigate the accuracy of the FD/PSF method. We consider three examples with solutions

(1) $u(x, y) = \sin(\pi x + \pi/4) \sin(\pi \cos y + \pi/4)$ and coefficients $a = b = 1$,

(2) $u(x, y) = \sin(\pi r(x)) \sin(\pi \cos y + \pi/4)$ and coefficients as in (4.4) for $\alpha = 1$,

(3) $u(x, y) = \cos(\pi r(x)) \sin(\pi \cos y + \pi/4)$ and coefficients as for example (2), but with Neumann boundary conditions at $r = 1$.

Here by $E2$ (resp. EM) we denote the absolute discretization error $\|u - u^h\|$ measured by the discrete l_2 (resp. maximum) norm. We count the number IT of

TABLE V

Convergence Factors ρ_W for Coefficients (4.4)

Relaxation	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$
SR	0.5254	0.5268	0.5270
NSR	0.4154	0.4155	0.4160
MRR	0.4670	0.4576	0.4714

TABLE VI

Convergence Factors ρ_W for Coefficients (4.4) and Neumann Boundary Conditions at $r = 1$ with First- and Second-Order Preconditioning

Relaxation	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$
First-order preconditioning			
SR	0.6526	0.6529	0.6522
NSR	0.6143	0.6129	0.6139
MRR	0.5880	0.5863	0.6047
Second-order preconditioning			
SR	0.5324	0.5332	0.5350
NSR	0.4667	0.4857	0.4921
MRR	0.4759	0.4770	0.4880

V-cycles needed in order to achieve an accuracy of $\| u_{IT}^h - u^h \| < E2$ for the ITth approximation u_{IT}^h. In all cases we started with the zero grid function. Now the convergence factor can be measured by the mean value [3], $\bar{\rho} = (\| u_{IT}^h - u^h \| / \| u^h \|)^{1/IT}$.

The corresponding convergence factor per work unit is written as $\bar{\rho}_W$. The numerical results are presented in Table VII. It can be seen that the accuracy is determined by the second-order FD method. In Table VIII we give the results for example (1) discretized by the fourth-order FD operator (2.4). The accuracy of about 10^{-7} on the finest grid is still determined by the FD method since the spectral accuracy lies about 10^{-12} (see [13]). But for almost all realistic applications, the fourth-order accuracy is acceptable.

The preceding numerical results demonstrate that combined FD and PSF problems can efficiently be solved by multigrid methods.

TABLE VII

Numerical Results for the Examples in Section 4.2 with Second-Order FDs

Example	M	N	EM	$E2$	IT	$\bar{\rho}_W$
1	32	8	8.95 (−2)	3.99 (−2)	1	0.3179
	64	16	1.06 (−3)	3.33 (−4)	2	0.3661
	128	32	2.35 (−4)	8.25 (−5)	3	0.3721
2	32	8	6.82 (−2)	1.63 (−2)	1	0.2964
	64	16	3.25 (−3)	1.51 (−3)	2	0.3371
	128	32	8.10 (−4)	3.77 (−4)	2	0.3517
3	32	8	7.51 (−2)	1.99 (−2)	2	0.4245
	64	16	5.51 (−3)	2.19 (−3)	3	0.4688
	128	32	1.35 (−3)	4.87 (−4)	3	0.5033

TABLE VIII

Numerical Results for Example (1) in Section 4.2 with Fourth-Order FDs

Example	M	N	EM	$E2$	IT	$\bar{\rho}_W$
1	32	8	8.55 (−2)	3.99 (−2)	1	0.3288
	64	16	1.20(−4)	4.78 (−5)	3	0.3761
	128	32	3.92 (−7)	1.57 (−7)	4	0.3804

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